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# Reduced-Basis Technique for Evaluating the Sensitivity Coefficients of the Nonlinear Tire Response

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**An efficient reduced-basis technique is presented for calculating the sensitivity of nonlinear tire response to variations in the design variables. The tire is discretized by using three-field mixed finite element models. The vector of structural response and its first- and second-order sensitivity coefficients (derivatives with respect to design variables) are each expressed as linear combinations of a small number of basis (or global approximation) vectors. The Bubnov-Galerkin technique is then used to approximate each of the finite element equations governing the response and the sensitivity coefficients, by a small number of algebraic equations in the amplitudes of these vectors. Path derivatives (derivatives of the response vector with respect to path parameters, e.g., load parameters) are used as basis vectors for approximating the response. A combination of the path derivatives and their derivatives with respect to the design variables is used for approximating the sensitivity coefficients. The potential of the proposed technique is discussed and its effectiveness is demonstrated by means of a numerical example of the Space Shuttle nose-gear tire subjected to uniform inflation pressure. The design variables are selected to be the material properties of the cord and rubber, as well as the cord diameters, end counts, and angles.**

## Nomenclature

$b_0, b_1, b_2, b_3$ = parameters used in defining the cord end counts (epi); see Eqs. (28) and Table 2
$\bar{b}_1, \bar{b}_2$ = geometric characteristics of the Space Shuttle nose-gear tire cross section; see Fig. 1
$d_1, d_2$ = cord diameters
$E_L, E_T$ = elastic moduli of the individual layers in the direction of the cords and normal to that direction, respectively
$E_{T_0}$ = value of $E_T$ at $s = \xi = 0$
$E_r, E_c$ = Young's moduli of the rubber and cord, respectively
$\{E\}$ = subvector of strain parameters for the shell model
epi = cord end count (ends per inch)
$G_{LT}, G_{TT}$ = shear moduli in plane of cords and normal to it
$G_r, G_c$ = shear moduli of the rubber and cord, respectively
$\{\hat{G}(\psi)\}$, $\{\bar{G}(\psi)\}$, $\{\bar{\bar{G}}(\psi, \bar{\psi})\}$ = vectors of nonlinear terms of the reduced systems; see Eqs. (7-9), (11) and (21)
$\{G(Z)\}$ = vector of nonlinear terms; see Eqs. (1)
$\{H\}$ = subvector of stress-resultant parameters for the shell model
$h$ = total thickness of the tire cross section
$h_0$ = thickness of the tire cross section at $\xi = 0$
$\bar{h}$ = $h/h_0$
$[K]$ = global linear structure matrix that includes linear stiffness and strain-displacement matrices; see Eqs. (1)
$[K_0]$ = linear stiffness matrix; see Eqs. (A2), Appendix A
$[\hat{K}]$, $[\bar{K}]$, $[\bar{\bar{K}}]$ = linear matrices of the reduced systems; see Eqs. (7-10), (13), and (17)
$M_s, M_\theta, M_{s\theta}$ = bending (and twisting) stress resultants
$\{M(X)\}$, $\{N(H, X)\}$ = subvectors of nonlinear terms; see Eqs. (A3), Appendix A
$N_s, N_\theta, N_{s\theta}$ = extensional stress resultants
$\{P\}$ = subvector of normalized applied loads; see Eqs. (A4), Appendix A
$p_0$ = intensity of normal pressure loading
$\{\hat{Q}\}$, $\{\bar{Q}\}$, $\{\bar{\bar{Q}}\}$ = load vectors of the reduced system; see Eqs. (7-9), (12), (16), and (22)
$\{Q\}$ = normalized load vector; see Eqs. (1)
$q$ = load parameter
$[R]$ = linear matrix containing products of integrals of shape functions; see Eqs. (A2), Appendix A
$r$ = number of basis vectors used in evaluating the response
$[S]$ = linear strain-displacement matrix; see Eqs. (A2), Appendix A
$s$ = meridional coordinate of the tire cross section
$U$ = total strain energy of the shell model
$u, v, w$ = displacement components of the reference surface of the shell in the meridional, circumferential, and normal directions
$\{X\}$ = subvector of nodal displacements of the shell model
$x_3$ = coordinate normal to the shell reference surface
$\{Z\}$ = response vector of the shell model, which includes the subvectors of strain parameters, stress-resultant parameters, and nodal displacements
$[\Gamma]$, $[\bar{\Gamma}]_I$, $[\bar{\bar{\Gamma}}]_I$ = matrices of basis vectors; see Eqs. (4-6), (23-25)
$\theta$ = circumferential (hoop) coordinate of the shell model
$\tilde{\theta}$ = cord angle with the $s$ axis; see Eq. (29)
$\theta_0, \theta_1, \theta_2$ = parameters used in defining $\tilde{\theta}$
$\lambda_I$ = material and geometric parameters of the tire constituents (material properties of

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the cord and rubber, as well as the cord diameters, end counts, and angles)

$\nu_{LT}$ = major Poisson's ratio of the individual layers

$\xi$ = dimensionless coordinate along the meridian of the tire cross section

$\phi_s$, $\phi_\theta$ = rotation components of the middle surface of the shell

$\{\psi\}$, $\{\bar{\psi}\}_l$, $\{\bar{\bar{\psi}}\}_l$ = vectors of undetermined coefficients of the reduced equations (amplitudes of the global approximation vectors); see Eqs. (4-6)

*Superscript*

$t$ = denotes matrix transposition

*Ranges of Indices*

$I$, $J$, $L$ = 1 to the total number of degrees of freedom (nodal displacements, strain parameters, and stress-resultant parameters)

$i$, $j$ = 1 to the total number of reduced degrees of freedom used in evaluating the sensitivity coefficients

$l$ = 1 to the total number of material and geometric parameters of the tire constituents

## Introduction

SIGNIFICANT advances have been made in the development of effective computational models and computational strategies for the numerical simulation of the nonlinear tire response (see, for example, Refs. 1-3). However, the use of nonlinear analysis in the automated optimum design of tires requires efficient techniques for calculating the sensitivity of the nonlinear tire response to variations in the design variables. The sensitivity coefficients (derivatives of the response vector with respect to design variables) are used to:

1) Determine a search direction in the direct application of nonlinear mathematical programming algorithms. When approximation concepts are used for the optimum design of tires, sensitivity derivatives are employed to construct explicit approximations for the critical and potentially critical behavior constraints.

2) Generate an approximation for the response of a modified tire (along with a reanalysis technique).

3) Assess the effects of uncertainties, in the material and geometric parameters of the computational model, on the tire response.

4) Predict the changes in the tire response due to changes in the parameters.

Two general procedures are currently used for calculating the sensitivity coefficients of the nonlinear structural response. The two approaches are (see, for example, Refs. 4-10) the direct differentiation, and adjoint variable methods. The first procedure is based on the implicit differentiation of the equations that describe the nonlinear structural response with respect to the desired parameters and the solution of the resulting sensitivity equations. In the adjoint variable method, an adjoint physical system is introduced whose solution permits the rapid evaluation of the desired sensitivity coefficients. Both procedures can be applied to either the governing discrete equations or the functional of the variational formulation of the structure (with a consequent change in the order of discretization and implicit differentiation).

The discrete models for an aircraft tire typically have a large number of degrees of freedom, and the calculation of the sensitivity coefficients of the nonlinear tire response can become quite expensive. Although efficient reduced-basis techniques have been developed for substantially reducing the number of degrees of freedom in the initial discretization and computational effort involved in the nonlinear analysis of tires (see, for example, Refs. 1, 2, and 11), no attempt has been made to adapt these techniques to the calculation of sensitivity coefficients. The present study attempts to fill this void. Specifically, this paper presents a reduced-basis technique and computational procedure for the efficient calculation of the first- and second-order sensitivity coefficients of the nonlinear tire response (derivatives of the tire response with respect to variations in the material and geometric parameters of the tire). The crux of the technique is the effective choice of basis vectors for approximating the sensitivity coefficients. Path derivatives (derivatives of the tire response vector with respect to a path parameter, for example, load parameter) are used as basis vectors for approximating the response. A combination of the path derivatives and their derivatives with respect to the material and geometric characteristics of the tire constituents (namely, the tire cord and rubber) is then used for approximating the sensitivity coefficients. Numerical results are presented for the case of uniform inflation pressure on the Space Shuttle orbiter nose-gear tire. Both the first- and second-order coefficients for the generalized displacements and stress resultants are given. The design variables are selected to be the material properties of the cord and rubber, as well as the cord diameters, end counts, and angles.

## Mathematical Formulation

### Governing Finite Element Equations

The tire is modeled by using a two-dimensional, moderate-rotation, laminated anisotropic shell theory with the effects of variation in material and geometric parameters included (Refs. 3 and 12). A total Lagrangian formulation is used for describing the deformation, and the fundamental unknowns consist of generalized displacements, strain components, and stress resultants of the tire. The governing finite element equations are obtained through the application of the three-field Hu-Washizu mixed variational principle. The governing finite element equations for the response vector and its first- and second-order sensitivity coefficients can be written in the following compact form:

$$[K]\{Z\} + \{G(Z)\} - q\{Q\} = 0 \tag{1}$$

$$\left[[K] + \left[\frac{\partial G_I}{\partial Z_J}\right]\right]\left\{\frac{\partial Z}{\partial \lambda}\right\}_l = -\left[\frac{\partial K}{\partial \lambda}\right]_l \{Z\} + q\left\{\frac{\partial Q}{\partial \lambda}\right\}_l \tag{2}$$

and

$$\left[[K] + \left[\frac{\partial G_I}{\partial Z_J}\right]\right]\left\{\frac{\partial^2 Z}{\partial \lambda^2}\right\}_l = -\left[\frac{\partial^2 K}{\partial \lambda^2}\right]_l \{Z\} - 2\left[\frac{\partial K}{\partial \lambda}\right]_l \left\{\frac{\partial Z}{\partial \lambda}\right\}_l - \left\{\frac{\partial^2 G_I}{\partial Z_J \partial Z_L}\frac{\partial Z_J}{\partial \lambda}\frac{\partial Z_L}{\partial \lambda}\right\}_l + q\left\{\frac{\partial^2 Q}{\partial \lambda^2}\right\}_l \tag{3}$$

where $[K]$ is the global linear structure matrix that includes "generalized" stiffness matrices of the tire; $\{Z\}$ is the response vector that includes strain parameters, stress-resultant parameters, and nodal displacements; $\{G(Z)\}$ is the vector of nonlinear terms; $q$ is a load parameter; $\{Q\}$ is the normalized applied load vector; $\lambda$ refers to a design variable, selected to be any of the material properties of the cord and rubber, cord diameters, end counts, and angles. The subscript $l$ in Eqs. (2) and (3) signifies the dependence of the vectors and matrices on the particular choice of the design variable. For convenience, the subscript $l$ is not, and will not be, used with the individual $\lambda$ in the equations. However, in succeeding discussions $\lambda_l$ is employed to denote a single design variable. The form of the arrays $[K]$, $\{G(Z)\}$, and $\{Q\}$ is described in Ref. 12. Note that in Eqs. (2) and (3), $\{G(Z)\}$ is assumed to be independent of $\lambda_l$, which represents a unique feature of the mixed formulation of the problem. Equations (1) are nonlinear in $\{Z\}$, but Eqs. (2) and (3) are linear in $\{\partial Z/\partial \lambda\}_l$ and $\{\partial^2 Z/\partial \lambda^2\}_l$, respectively.

**Basis Reduction and Reduced System of Equations**

The response vector $\{Z\}$ and its first- and second-order derivatives with respect to $\lambda_l$, $\{\partial Z/\partial\lambda\}_l$, and $\{\partial^2 Z/\partial\lambda^2\}_l$ are each expressed as a linear combination of a few preselected basis vectors. The approximations can be expressed by the following transformations:

$$\{Z\} = [\Gamma]\{\psi\} \tag{4}$$

$$\left\{\frac{\partial Z}{\partial\lambda}\right\}_l = [\bar{\Gamma}]_l\{\bar{\psi}\}_l \tag{5}$$

and

$$\left\{\frac{\partial^2 Z}{\partial\lambda^2}\right\}_l = [\bar{\bar{\Gamma}}]_l\{\bar{\bar{\psi}}\}_l \tag{6}$$

The columns of the matrices $[\Gamma]$, $[\bar{\Gamma}]_l$, and $[\bar{\bar{\Gamma}}]_l$ in Eqs. (4–6) are the basis vectors; and the elements of the vectors $\{\psi\}$, $\{\bar{\psi}\}_l$, $\{\bar{\bar{\psi}}\}_l$ are the amplitudes of the approximation vectors that are, as yet, unknowns. Note that the number of basis vectors in Eqs. (4–6) is considerably smaller than the total number of degrees of freedom (components of the vectors $\{Z\}$, $\{\partial Z/\partial\lambda\}_l$, and $\{\partial^2 Z/\partial\lambda^2\}_l$).

A Bubnov-Galerkin technique is now used to replace the governing equations for the tire response and its first- and second-order sensitivity coefficients, Eqs. (1–3), by the following reduced equations in the unknowns $\{\psi\}$, $\{\bar{\psi}\}_l$, $\{\bar{\bar{\psi}}\}_l$:

$$[\hat{K}]\{\psi\} + \{\hat{G}(\psi)\} - q\{\hat{Q}\} = 0 \tag{7}$$

$$\left[[\bar{K}] + \left[\frac{\partial \bar{G}_i}{\partial \psi_j}\right]\right]\{\bar{\psi}\}_l = -\left[\frac{\partial \bar{K}}{\partial\lambda}\right]_l\{\psi\} + q\left\{\frac{\partial \bar{Q}}{\partial\lambda}\right\}_l \tag{8}$$

$$\left[[\bar{\bar{K}}] + \left[\frac{\partial \bar{\bar{G}}_i}{\partial \psi_j}\right]\right]\{\bar{\bar{\psi}}\}_l = -\left[\frac{\partial^2 \bar{\bar{K}}}{\partial\lambda^2}\right]_l\{\psi\} - 2\left[\frac{\partial \bar{\bar{K}}}{\partial\lambda}\right]_l\{\bar{\psi}\}_l - \{\bar{\bar{G}}(\psi,\bar{\psi})\}_l + q\left\{\frac{\partial^2 \bar{\bar{Q}}}{\partial\lambda^2}\right\}_l \tag{9}$$

where

$$[\hat{K}] = [\Gamma]^t[K][\Gamma] \tag{10}$$

$$\{\hat{G}(\psi)\} = [\Gamma]^t\{G(\psi)\} \tag{11}$$

$$\{\hat{Q}\} = [\Gamma]^t\{Q\} \tag{12}$$

$$[\bar{K}] = [\bar{\Gamma}]_l^t[K][\bar{\Gamma}]_l \tag{13}$$

$$\left[\frac{\partial \bar{G}_i}{\partial \psi_j}\right] = [\bar{\Gamma}]_l^t\left[\frac{\partial G_I}{\partial Z_J}\right][\bar{\Gamma}]_l \tag{14}$$

$$\left[\frac{\partial \bar{K}}{\partial\lambda}\right]_l = [\bar{\Gamma}]_l^t\left[\frac{\partial K}{\partial\lambda}\right]_l[\Gamma] \tag{15}$$

$$\left\{\frac{\partial \bar{Q}}{\partial\lambda}\right\}_l = [\bar{\Gamma}]_l^t\left\{\frac{\partial Q}{\partial\lambda}\right\}_l \tag{16}$$

$$[\bar{\bar{K}}] = [\bar{\bar{\Gamma}}]_l^t[K][\bar{\bar{\Gamma}}]_l \tag{17}$$

$$\left[\frac{\partial \bar{\bar{G}}_i}{\partial \psi_j}\right] = [\bar{\bar{\Gamma}}]_l^t\left[\frac{\partial G_I}{\partial Z_J}\right][\bar{\bar{\Gamma}}]_l \tag{18}$$

$$\left[\frac{\partial^2 \bar{\bar{K}}}{\partial\lambda^2}\right]_l = [\bar{\bar{\Gamma}}]_l^t\left[\frac{\partial^2 K}{\partial\lambda^2}\right]_l[\Gamma] \tag{19}$$

$$\left[\frac{\partial \bar{\bar{K}}}{\partial\lambda}\right]_l = [\bar{\bar{\Gamma}}]_l^t\left[\frac{\partial K}{\partial\lambda}\right]_l[\bar{\Gamma}]_l \tag{20}$$

$$\{\bar{\bar{G}}(\psi,\bar{\psi})\}_l = [\bar{\bar{\Gamma}}]_l^t\{G(\psi,\bar{\psi})\}_l \tag{21}$$

$$\left\{\frac{\partial^2 \bar{\bar{Q}}}{\partial\lambda^2}\right\}_l = [\bar{\bar{\Gamma}}]_l^t\left\{\frac{\partial^2 Q}{\partial\lambda^2}\right\}_l \tag{22}$$

Note that Eqs. (7) are nonlinear in the unknowns $\{\psi\}$ and Eqs. (8) and (9) are linear in $\{\bar{\psi}\}_l$ and $\{\bar{\bar{\psi}}\}_l$, respectively. The vector $\{G(\psi)\}$ in Eqs. (11) is obtained by replacing $\{Z\}$ in the vector $\{G(Z)\}$ by its expression in terms of $\{\psi\}$, Eqs. (4). The vector $\{G(\psi,\bar{\psi})\}_l$ in Eqs. (21) is obtained by replacing $\{Z\}$ and $\{\partial Z/\partial\lambda\}_l$ in the vector $\{(\partial^2 G_I/\partial Z_J\partial Z_L)(\partial Z_J/\partial\lambda)(\partial Z_L/\partial\lambda)\}_l$ by their expressions in terms of $\{\psi\}$ and $\{\bar{\psi}\}_l$, Eqs. (4) and (5).

**Selection and Generation of Basis Vectors**

The effectiveness of the proposed technique for calculating the sensitivity coefficients depends, to a great extent, on the proper choice of the basis vectors (the columns of the matrices $[\Gamma]$, $[\bar{\Gamma}]_l$, and $[\bar{\bar{\Gamma}}]_l$). An effective choice for the basis vectors used in approximating the response vector $\{Z\}$, Eqs. (4), was found to be the various-order path derivatives (derivatives with respect to the load parameter $q$); that is, the matrix $[\Gamma]$ used in approximating $\{Z\}$, over a range of values of $q$, includes the response vector corresponding to a particular value of $q$ (viz. $q^0$) and its various-order derivatives with respect to $q$, evaluated at the same value of $q^0$, or

$$[\Gamma] = \left[\{Z\}\left\{\frac{\partial Z}{\partial q}\right\}\left\{\frac{\partial^2 Z}{\partial q^2}\right\}\left\{\frac{\partial^3 Z}{\partial q^3}\right\}\cdots\right]_{q^0} \tag{23}$$

The basis vectors for approximating the sensitivity coefficients $\{\partial Z/\partial\lambda\}_l$ and $\{\partial^2 Z/\partial\lambda^2\}_l$ were selected to be a combination of 1) the various-order path derivatives, that is, the columns of the matrix $[\Gamma]$; 2) the first derivatives of $[\Gamma]$ with respect to $\lambda_l$, that is, $[\partial\Gamma/\partial\lambda]_l$; and in the case of $\{\partial^2 Z/\partial\lambda^2\}_l$; 3) the second derivatives of $[\Gamma]$ with respect to $\lambda_l$, that is, $[\partial^2\Gamma/\partial\lambda^2]_l$. Therefore:

$$[\bar{\Gamma}]_l = \left[[\Gamma]\left[\frac{\partial\Gamma}{\partial\lambda}\right]_l\right] \tag{24}$$

$$[\bar{\bar{\Gamma}}]_l = \left[[\Gamma]\left[\frac{\partial\Gamma}{\partial\lambda}\right]_l\left[\frac{\partial^2\Gamma}{\partial\lambda^2}\right]_l\right] \tag{25}$$

The number of basis vectors in $[\bar{\Gamma}]_l$ and $[\bar{\bar{\Gamma}}]_l$, for each $\lambda$, is two and three times, respectively, the number of basis vectors in $[\Gamma]$.

The rationale for the particular choice of $[\bar{\Gamma}]_l$ and $[\bar{\bar{\Gamma}}]_l$ is based on the following two facts:

1) Differentiating Eqs. (4) with respect to $\lambda$ leads to

$$\left\{\frac{\partial Z}{\partial\lambda}\right\}_l = [\Gamma]\left\{\frac{\partial\psi}{\partial\lambda}\right\}_l + \left[\frac{\partial\Gamma}{\partial\lambda}\right]_l\{\psi\} \tag{26}$$

and

$$\left\{\frac{\partial^2 Z}{\partial\lambda^2}\right\}_l = [\Gamma]\left\{\frac{\partial^2\psi}{\partial\lambda^2}\right\}_l + 2\left[\frac{\partial\Gamma}{\partial\lambda}\right]_l\left\{\frac{\partial\psi}{\partial\lambda}\right\}_l + \left[\frac{\partial^2\Gamma}{\partial\lambda^2}\right]_l\{\psi\} \tag{27}$$

that is, the expression for $\{\partial Z/\partial\lambda\}_l$ includes *both* $[\Gamma]$ and $[\partial\Gamma/\partial\lambda]_l$; the expression for $\{\partial^2 Z/\partial\lambda^2\}_l$ includes $[\Gamma]$, $[\partial\Gamma/\partial\lambda]_l$, and $[\partial^2\Gamma/\partial\lambda^2]_l$.

2) The use of free parameters $\{\bar{\psi}\}_l$ and $\{\bar{\bar{\psi}}\}_l$, instead of the fixed amplitudes $\{\psi\}$, $\{\partial\psi/\partial\lambda\}_l$, and $\{\partial^2\psi/\partial\lambda^2\}_l$ in Eqs. (26) and (27), is expected to improve the accuracy of the approximations for $\{\partial Z/\partial\lambda\}_l$ and $\{\partial^2 Z/\partial\lambda^2\}_l$, over a wide range of $q$. The free parameters $\{\bar{\psi}\}_l$ and $\{\bar{\bar{\psi}}\}_l$ are obtained by applying the Bubnov-Galerkin technique to Eqs. (2) and (3), resulting in Eqs. (8) and (9).

The path derivatives (columns of the matrix $[\Gamma]$) are obtained by successive differentiation of the governing finite element equations of the tire, Eqs. (1), with respect to the parameter $q$. The recursion relations for evaluating the path derivatives are given in Ref. 13. Note that *only one matrix factorization is needed for generating all the path derivatives.*

The derivatives of $[\Gamma]$ with respect to $\lambda_l$, $[\partial\Gamma/\partial\lambda]_l$, and $[\partial^2\Gamma/\partial\lambda^2]_l$ are obtained by differentiating each of the recursion relations for evaluating the path derivatives with respect to $\lambda_l$. The resulting equations have the same left-hand sides as those of the original recursion relations, and therefore, *no additional matrix factorizations are needed for generating* $[\partial\Gamma/\partial\lambda]_l$ *and* $[\partial^2\Gamma/\partial\lambda^2]_l$.

## Computational Procedure

The computational procedure for generating the nonlinear response vector $\{Z\}$ and its sensitivity coefficients $\{\partial Z/\partial\lambda\}_l$ and $\{\partial^2 Z/\partial\lambda^2\}_l$ can be conveniently divided into two distinct phases, namely: 1) evaluation of the basis vectors, at a particular value of $q$ (viz. $q^0$) and generation of the reduced equations; and 2) marching with the reduced equations in the solution space and generating the response and sensitivity coefficients at different values of $q$. For each value of $q$, the vector of reduced unknowns $\{\psi\}$ is obtained by solving the reduced nonlinear equations, Eqs. (7). Then the vectors $\{\bar{\psi}\}_l$ and $\{\bar{\bar{\psi}}\}_l$ associated with the same value of $\{\psi\}$ are evaluated by solving the reduced linear equations, Eqs. (8) and (9). The response vector and its sensitivity coefficients $\{Z\}$, $\{\partial Z/\partial\lambda\}_l$, and $\{\partial^2 Z/\partial\lambda^2\}_l$ are obtained by using Eqs. (4–6). The process is repeated for different values of $q$.

## Numerical Studies

To test and evaluate the effectiveness of the proposed reduced-basis technique, the sensitivity coefficients of the nonlinear response of the Space Shuttle nose-gear tire were generated by this technique. Comparisons were made with the sensitivity coefficients obtained by using the full system of equations of the finite element model. Typical results are presented in Figs. 3–7 for the case of uniform pressure $p_0$ on the Space Shuttle nose-gear tire. The geometric and material characteristics of the tire are given in Table 1 and Fig. 1. Because of symmetry, only half the tire cross section was modeled. The tire cross section was divided into seven segments as shown in Fig. 2. Each segment contained a different number of layers, different material properties corresponding to different cord content in the composite, and varying cord orientations (see Refs. 3 and 14). Spline interpolation was used to smooth the experimental data that were used to define the geometric and material characteristics of the two-dimensional shell model (see Refs. 3 and 14). The outer surface of the tire was chosen to be the reference surface for the shell model.

The numerical studies were performed using three-field mixed finite element models for the discretization of the tire in the meridional direction. Linear interpolation functions are used for approximating each of the stress resultants and strain components, and quadratic Lagrangian interpolation functions are used for approximating each of the generalized displacements. The integrals in the governing equations are evaluated using a two-point Gauss-Legendre numerical quadrature formula. Sixty finite elements were used in modeling half the tire cross section for a total of 960 strain parameters, 960 stress-resultant parameters, and 560 nonzero generalized displacements. The three displacement components in the region $0.45 \leq \xi \leq 0.5$ are totally restrained and, in addition, the rotation components are restrained at $\xi = 0.5$.

The cord end counts, epi, and cord angles with the meridional direction $\tilde{\theta}$ were approximated by the following formulas:

$$\text{epi} = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 \tag{28}$$

$$\tilde{\theta} = \theta_0 - \theta_1\xi - \theta_2\xi^2 \tag{29}$$

where $\theta_0 = 54.38$, $\theta_1 = 3.884$, $\theta_2 = 148.96$, and the numerical values of $b_0$, $b_1$, $b_2$, and $b_3$ are given in Table 2.

Figure 3 shows plots of the loading vs the total strain energy $U$, as well as its first- and second-order sensitivity coefficients with respect to the cord diameters $d_1$ and $d_2$, and the material parameters $E_c$, $E_r$, $G_c$, and $G_r$, where subscripts $r$ and $c$ refer to rubber and cord, respectively. As can be seen from Fig. 3, $U$ is more sensitive to variations in $d_2$ and $E_c$ than to the other cord diameter and material properties.

Figure 4 shows plots for the loading vs the first- and second-order sensitivity coefficients with respect to the parameters $\theta_0$, $\theta_1$, $\theta_2$ and $b_0$, $b_1$, $b_2$, $b_3$ for region 2. As can be seen from Fig. 4, the nonlinear tire response is considerably more sensitive to variations in $b_0$ and $\theta_0$ than all other $b$ and $\theta$.

The basis vectors for both the response and sensitivity coefficients were generated for the unloaded tire, $p_0 = 0$, and were thus obtained by solving a linear system of finite element equations. Then the reduced equations for evaluating the response and sensitivity coefficients were generated. The basis vectors were not updated throughout the range of loading considered. An indication of the accuracy of $w_c$ and $U$, as well as first- and second-order sensitivity coefficients obtained by the reduced-basis technique, is given in Figs. 5 and 6.

For the range of loading considered, $w_c$ and $U$ obtained by using 15 basis vectors are almost indistinguishable from those obtained by using the full system of finite element equations. The first- and second-order sensitivity coefficients (first- and second-order derivatives of $U$ with respect to $d_2$, $E_r$, $E_c$, $\theta_0$,

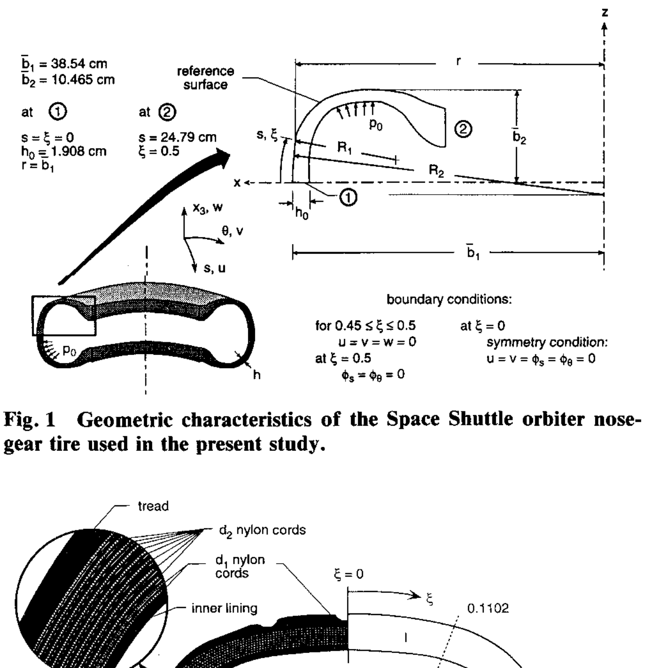


**Fig. 1 Geometric characteristics of the Space Shuttle orbiter nose-gear tire used in the present study.**

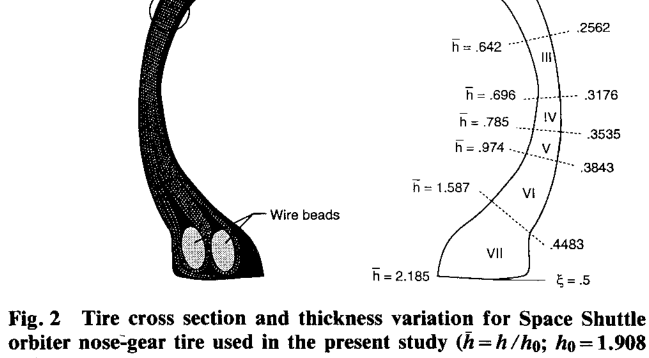


**Fig. 2 Tire cross section and thickness variation for Space Shuttle orbiter nose-gear tire used in the present study ($\bar{h} = h/h_0$; $h_0 = 1.908$ cm).**

**Table 1 Values of elastic constants of tire constituents used in present study**

| Tire constituent | Young's modulus, $E$(Pa) | Shear modulus, $G$(Pa) | Poisson's ratio |
|---|---|---|---|
| Rubber | $3.10 \times 10^6$ | $1.04 \times 10^6$ | 0.49 |
| Nylon cord | $2.41 \times 10^9$ | $4.83 \times 10^6$ | 0.66 |
| Bead[a] | $2.00 \times 10^{11}$ | $7.69 \times 10^{10}$ | 0.30 |

[a]Since the deformations are small in the bead area, it is reasonable to assume that the bead wires are isotropic.

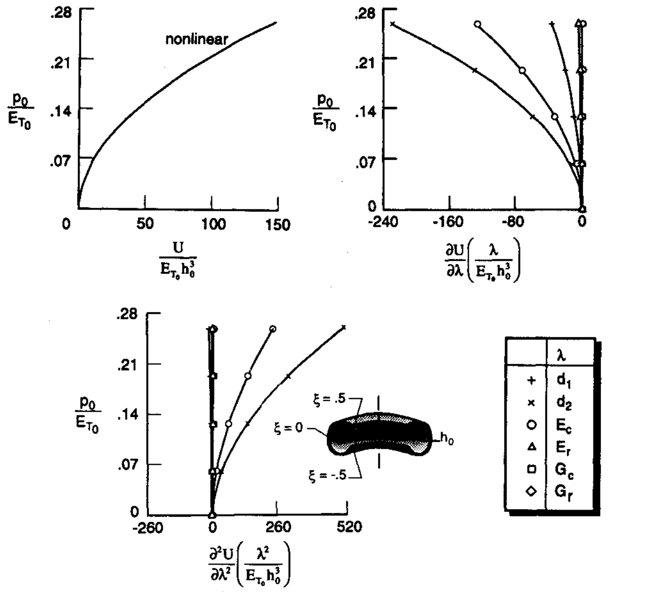


**Fig. 3 Nonlinear response and sensitivity coefficients with respect to cord diameters and material properties. Space Shuttle orbiter nose-gear tire subjected to uniform inflation pressure. ($p_0 = 2.068$ MPa, $E_{T_0} = 8.0$ MPa, $h_0 = 1.908$ cm.) Results from full system of equations.**

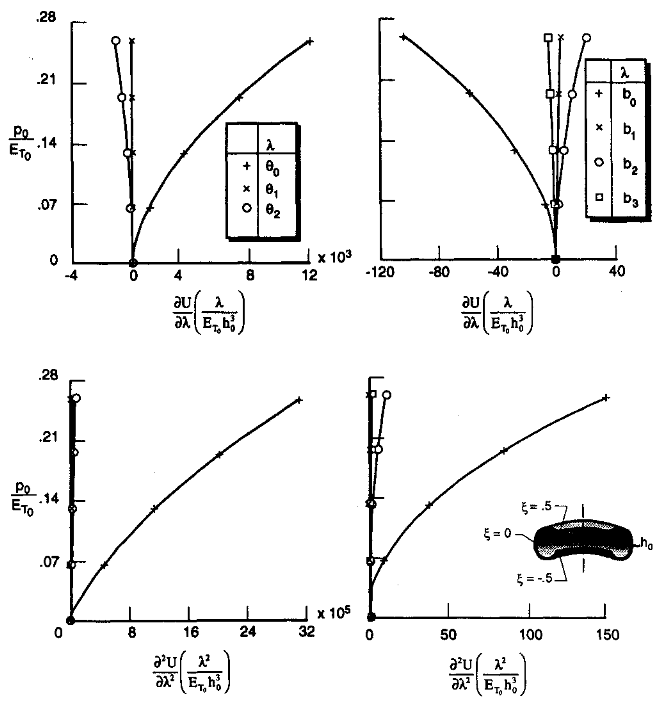


**Fig. 4 First- and second-order sensitivity coefficients of the tire response with respect to parameters of cord angle [see Eqs. (29)] and cord end count in region 2 [see Eqs. (28) and Table 2]. Space Shuttle orbiter nose-gear tire subjected to uniform inflation pressure. ($p_0 = 2.068$ MPa, $E_{T_0} = 8.0$ MPa, $h_0 = 1.908$ cm.)**

and $b_0$ of region 2), obtained by using 30 and 45 basis vectors, are as accurate as $U$ obtained by using 15 basis vectors ($r = 15$). For the first-order sensitivity coefficients, the 30 basis vectors consist of the first 15 derivatives of $\{Z\}$ with respect to $p_0$, $\{\partial^{s'}Z/\partial q^{s'}\}$ ($s' = 1$ to 15), and their first-order derivatives with respect to $\lambda_l$. For the second-order sensitivity coefficients, the 45 basis vectors consist of the 30 basis vectors used in approximating the first-order sensitivity coefficients and the second derivatives of $\{\partial^{s'}Z/\partial q^{s'}\}$ with respect to $\lambda_l$.

An indication of the accuracy of the meridional distributions of the displacements $u$, $v$, $w$, and their second-order derivatives with respect to $d_1$ and $d_2$ obtained by the reduced-basis technique is given in Fig. 7. The displacements obtained by using 15 basis vectors are almost indistinguishable from those obtained by solving the full system of finite element equations. The high accuracy of the second-order sensitivity coefficients predicted by the reduced-basis technique with $r = 15$ is clearly demonstrated in Fig. 7.

The computational time associated with the foregoing technique is considerably less than that associated with the direct application of Eqs. (2) and (3). This is particularly true when the sensitivity coefficients are needed at several different values of $p_0$. This is because the decomposed full-structure matrix on the left-hand-sides of Eqs. (2) and (3) is needed for each value of $p_0$ at which the sensitivity coefficients are required. Unless the response vector $\{Z\}$ is obtained by solving the full system of nonlinear equations, Eqs. (1), by using the Newton-Raphson technique the decomposed left-hand-side matrix of Eqs. (2) and (3) is not readily available (which is the case when the response vector is obtained by using either the quasi-Newton method or reduced-basis technique). By contrast, in the foregoing technique the most time-consuming operations are those associated with the generation of the basis vectors and reduced equations. The computational time expended in the solution of the reduced equations for the response and sensitivity coefficients is very small. Since the basis vectors are evaluated at $p_0 = 0$, their generation requires only the decomposition of the linear matrix of the tire model $[K]$, evaluation of the right-hand sides of the recursion relations for the basis vectors, and forward-reduction/back-substitution. These operations are computationally less expensive than the decompositions of the left-hand-side matrices of Eqs. (2) and (3). For the present problem, the computer time required to generate the sensitivity coefficients using Eqs. (2) and (3) at different values of $p_0$, after the response vectors $\{Z\}$ were obtained, by using the reduced-basis technique, was over four times that of the foregoing technique. This ratio increases rapidly as the number of times of evaluating the sensitivity coefficients increases.

## Potential of the Foregoing Reduced-Basis Technique

The foregoing reduced-basis technique appears to have high potential for use in the automated analysis and design of tires.

**Table 2 Numerical values of the coefficients used in approximating the end count for the Space Shuttle nose-gear tire used in the present study [see Eq. (28)]**

| Region[a] | $b_0$ | $b_1$ | $b_2$ | $b_3$ |
|---|---|---|---|---|
| 1 | 29.91 | 5.577 | −121.3 | 155.6 |
| 2 | 22.74 | −2.555 | −62.96 | 62.55 |
| 3 | 18.23 | −1.502 | −76.78 | 128.43 |

[a]Regions 1 and 3 refer to the bottom and top two layers of the shell model (which have cords); Region 2 refers to all the other layers with cords.

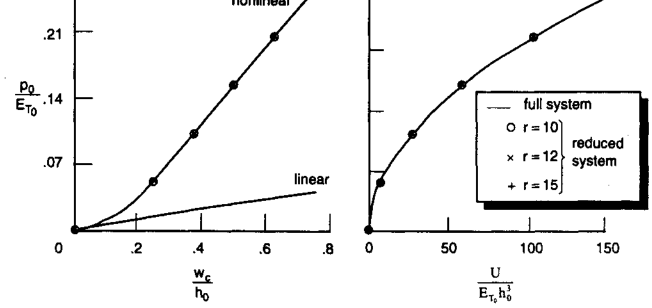


**Fig. 5 Accuracy of tire response obtained by the reduced-basis technique. Space Shuttle orbiter nose-gear tire subjected to uniform inflation pressure. ($p_0 = 2.068$ MPa, $E_{T_0} = 8.0$ MPa, $h_0 = 1.908$ cm.) Point $c$ corresponds to $\xi = 0$.**

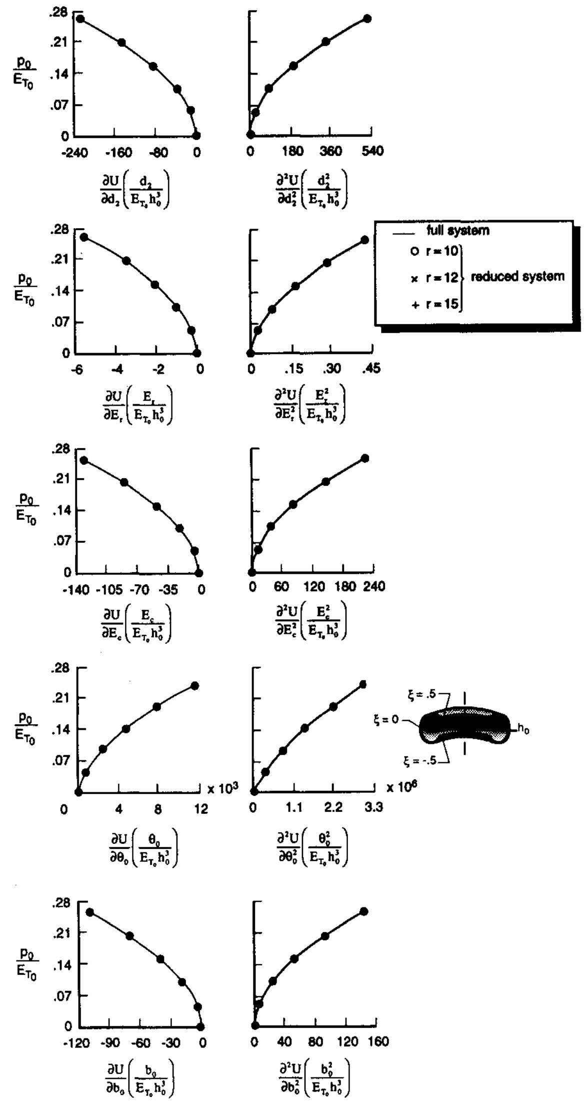


Fig. 6 Accuracy of first- and second-order sensitivity coefficients of tire response obtained by the reduced-basis technique. Space Shuttle orbiter nose-gear tire subjected to uniform inflation pressure. ($p_0 = 2.068$ MPa, $E_{T_0} = 8.0$ MPa, $h_0 = 1.908$ cm.)

The numerical studies conducted clearly demonstrate the accuracy and effectiveness of the technique. In particular, the following comments seem to be in order:

1) The particular choice of the basis vectors used herein for approximating the first- and second-order sensitivity coefficients (columns of the matrices $[\bar{\Gamma}]_l$ and $[\bar{\bar{\Gamma}}]_l$) allows the accurate prediction of these derivatives for a wide range of values of the control parameters.

2) The computational procedure developed for predicting the nonlinear response using the reduced-basis technique (see, for example, Ref. 13) can now be extended to the prediction of the sensitivity derivatives as well. When a new set of basis vectors for approximating the response is generated, the derivatives of each of these vectors with respect to the design variables (columns of the matrices $[\partial\Gamma/\partial\lambda]_l$ and $[\partial^2\Gamma/\partial\lambda^2]_l$) are generated as well.

3) The computational time associated with the foregoing technique is considerably less than the direct application of Eqs. (2) and (3). This is particularly true when the reduced-basis technique is used in generating the nonlinear response and the sensitivity coefficients are needed at several different values of the load parameter.

4) The foregoing technique can be easily extended to the evaluation of the cross (mixed) second-order derivatives, as well as structural reanalysis of tires (analysis of modified tires). For the cross second-order derivatives, the matrix of basis vectors, Eqs. (25), includes the derivatives with respect to more than one design variable. However, to keep the number of basis vectors small, only a few of the cross derivatives should be used at any one time. In structural reanalysis, the set of basis vectors included in $[\bar{\Gamma}]_l$ and $[\bar{\bar{\Gamma}}]_l$ are used for predicting the nonlinear response of the modified structure.

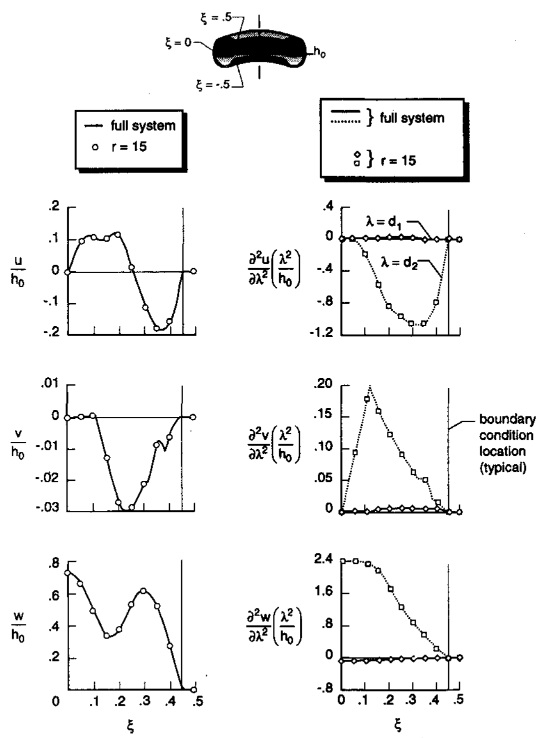


Fig. 7 Accuracy of the meridional variations of the displacements and their second-order sensitivity coefficients obtained by the reduced-basis technique. Space Shuttle orbiter nose-gear tire subjected to uniform inflation pressure. ($p_0 = 2.068$ MPa, $E_{T_0} = 8.0$ MPa, $h_0 = 1.908$ cm.)

## Concluding Remarks

An efficient reduced-basis technique is presented for calculating the sensitivity of nonlinear tire response to variations in the design variables. The tire is modeled by using a two-dimensional, moderate-rotation, laminated anisotropic shell theory with the effects of variation in material and geometric parameters included. A total Lagrangian formulation is used for describing the deformation and the fundamental unknowns consist of the generalized displacements, strain components, and stress resultants of the tire. The governing finite-element equations are obtained through the application of the three-field Hu-Washizu mixed variational principle.

The vector of structural response and its first- and second-order sensitivity coefficients (derivatives with respect to design variables) are each expressed as a linear combination of a small number of basis (or global approximation) vectors. The Bubnov-Galerkin technique is then used to approximate each of the finite element equations governing the response, and the sensitivity coefficients, by a small number of algebraic equations in the amplitudes of these vectors. The path derivatives (derivatives of the response vector with respect to a path parameter) are used as basis vectors for approximating the response. A combination of the path derivatives and their derivatives with respect to the design variables is used for approximating the sensitivity coefficients.

The potential of the proposed technique is discussed and the effectiveness of the basis vectors used in approximating the sensitivity coefficients is demonstrated by means of a numerical example of the Space Shuttle nose-gear tire subjected to uniform inflation pressure. The design variables are selected to be the material properties of the cord and rubber, as well as the cord diameters, end counts, and angles.

## Appendix A: Form of the Arrays in the Governing Discrete Equations of the Tire

The governing discrete equations of the tire, Eqs. (1), consist of the relations between the stress resultants and strain components of the tire, strain-displacement relations, and equilibrium equations of the tire (see Ref. 1). The response vector $\{Z\}$ can be partitioned into the subvectors of strain parameters $\{E\}$, stress-resultant parameters $\{H\}$, and nodal displacements $\{X\}$ as follows:

$$\{Z\} = \begin{Bmatrix} E \\ H \\ X \end{Bmatrix} \tag{A1}$$

The different arrays in Eqs. (1) can be partitioned as follows:

$$[K] = \begin{bmatrix} K_0 & R & 0 \\ R^t & 0 & S \\ 0 & S^t & 0 \end{bmatrix} \tag{A2}$$

$$\{G(Z)\} = \begin{Bmatrix} 0 \\ M(X) \\ N(H, X) \end{Bmatrix} \tag{A3}$$

$$\{Q\} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} \tag{A4}$$

where $[K_0]$ is a linear stiffness matrix; $[R]$ is a linear matrix containing integrals of products of shape functions; $[S]$ is the linear strain-displacement matrix; $\{M(X)\}$ and $\{N(H, X)\}$ are the subvectors of nonlinear terms; $\{P\}$ is the subvector of normalized applied loads; 0 refers to a null matrix or vector; and superscript $t$ denotes transposition. Note that the strain components and stress resultants are allowed to be discontinuous at interelement boundaries, and therefore, can be eliminated on the element level.

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